THE RESPONSE OF A FINITE LENGTH TUBE OF VISCOUS COMPRESSIBLE FLUID TO A STEP FUNCTION IN PRESSURE

A THESIS

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LIST OF SYMBOLS

A	Dimensionless Parameter
B	Dimensionless Parameter
D	Diameter
f	Friction Factor
G	Mass Flow per Unit Area
$\overline{G}$	Non-dimensional Mass Flow per Unit Area
G_n	Denotes Mass Flow per Unit Area at Station n
L	Length
N	Non-dimensional Time
ΔN	Non-dimensional Time Increment
P	Pressure
$\overline{P}$	Non-dimensional Pressure
P_n	Denotes Pressure at Station n
Q	Non-dimensional Parameter
R	Gas Constant
Rey	Reynolds Number
t	Time
$\overline{t}$	Non-dimensional Time
Δt	Time Increment
T	Absolute Temperature
V	Velocity
X	Spatial Co-ordinate

$\bar{X}$	Non-dimensional Spatial Co-ordinate
ΔX	Spatial Co-ordinate Increment
η	Non-dimensional Parameter
μ	Viscosity
ρ	Density

SUMMARY

Problems which can be described by non-linear partial differential equations are frequently encountered in unsteady flow. The assumption of quasi-steady flow is often made to simplify the solution of these equations. However, the conditions which justify the assumption of quasi-steady flow are not well defined. This study was made to help define the magnitude of the time increment necessary to justify the assumption of quasi-steady flow.

A study was conducted to determine the pressure distribution along a tube as a function of time, following a step input in pressure. A viscous, compressible fluid was considered, enclosed in a smooth tube, and the assumption of isothermal one-dimensional flow was made. With the aid of this assumption, the following equations were derived which govern the resulting flow:

$$G^2 = \frac{(P/RT)\ \frac{\partial P}{\partial x}}{(1/P)\ \frac{\partial P}{\partial x} - 4f/2D}$$

$$\frac{\partial G}{\partial x} = -\ (1/RT)\ \frac{\partial P}{\partial t}$$

The friction factor, f , was next expressed in terms of the mass flow per unit area, G , using Nikuradse's smooth pipe data for turbulent flow, and $4f = 64/Rey$ for laminar flow, where Rey is the Reynolds

Number. The flow is treated as either laminar or turbulent by using the expression for mass flow per unit area which incorporates the appropriate friction factor. For the boundary and initial conditions used, the flow was primarily turbulent.

The equations were non-dimensionalized, yielding two essential similarity parameters, L/D, and $Q = L^2 P_o^2 / \mu^2 RT$, where L is the length, D the diameter, P_o is a reference pressure, μ is the coefficient of viscosity, R is the gas constant, and T is the absolute temperature. The resulting partial differential equations were approximated by finite difference equations.

These finite difference equations were then solved on the IBM 650, using a range of the governing parameters, to give the pressure and mass flow as a function of distance along the tube and of time.

The state of development of the flow was described by expressing the difference between the mass flows at the tube ends as a percentage of deviation from the steady state mass flow. Approximate relationships were then graphically presented for the time required for a flow with some given initial boundary conditions to develop its steady state pressure profile, as a function of the initial conditions.

Analysis of the numerical solutions disclosed that the time required for convergence was only a weak function of the pressure level. A new non-dimensional quantity which was then defined, was proportional to true time required for convergence. Finally, this new quantity was graphically presented as a function of the boundary conditions.

From this data, it was concluded that the time required for the development of steady state flow is a strong function of the initial pressure distribution and the L/D ratio of the geometry, and a weak function of the pressure level.

These functions are presented graphically. From this data, the time required for the convergence of any flow within the range of conditions considered in this study may be estimated. The assumption of quasi-steady flow is thought to be justified in an unsteady flow problem if the time increment being considered is greater than the step function convergence times studied in this thesis.

CHAPTER I

INTRODUCTION

The measurement of transient pneumatic pressures by means of a length of constant area tubing attached to a sensing unit has been the subject of numerous investigations during the past twenty-five years[1, 2, 3*]. The early attempts to predict pneumatic lag were concerned primarily with the measurement of pitot tube static pressures for aircraft whose speeds lay in the 200-400 mph range. The transients encountered in this velocity regime were small in nature so that the pneumatic lag could be predicted by a first order linear differential equation.

The introduction of transonic and supersonic vehicles renewed interest in the pneumatic lag problem in that it could no longer be predicted to a satisfactory degree of accuracy by a first order linear differential equation. The equation of motion, Equation (2.7), for an isothermal, one-dimensional, viscous, compressible flow in a tube subjected to varying end pressures is a non-linear partial differential equation. Because of the difficulty of solution of this equation, the assumption of quasi-steady flow is often made as a simplifying assumption[4, 5, 6]. This assumption says that the mass flow through a tube instantaneously assumes its steady state value, corresponding to the instantaneous end pressures on the tube, and implies that inertia of

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[*] Superscripts refer to items in Bibliography

the gases can be neglected, and that steady state friction effects are applicable. In effect, this assumption of quasi-steady flow uncouples the effects of time and distance along the tube and results in a non-linear ordinary differential equation. This assumption appears to give good results in many situations[1, 2, 3], but it is of interest to try to predict its limitations.

The object of this study is to find the time required for steady state mass flow to develop in a tube which is initially at constant pressure throughout, and is suddenly subjected to a step function in pressure at one end.

To accomplish this object, several specific cases are solved with the aid of a digital computer, and the results non-dimensionalized so that the time required for the development of steady state flow in any system whose parameters fall within the range of those considered can be estimated.

If the order of magnitude of the time increment being considered in a specific problem is greater than that time increment obtained from this data, the assumption of quasi-steady flow should be justified.

CHAPTER II

THEORY

Derivation of Basic Equations.--The problem being considered is a transient one. However, if the finite length tube is divided into small units of length, and the assumption of quasi-steady flow is made within each of these small lengths, a great simplification is effected in the solution of the governing equations. The justification of this assumption is that in the limit, as the unit length becomes infinitesimally small, the mass flow within the unit length becomes a function of time alone, and not of position within the unit length.

The steady flow momentum equation can be written for one-dimensional flow in differential form as[7],

$$dp + (1/2)\rho V^2(4f dx/D) + \rho V^2 dV/V = 0 \qquad (2.1)$$

where p is the pressure, V is the velocity, f is the friction factor, defined as the ratio of the wall shearing stress to the dynamic head of the stream, ρ is the density, x is the distance along the tube and D is the tube diameter. It should be noted that inertia effects are being neglected by the use of Equation (2.1). Next, the continuity equation is introduced:

$$\rho V = G \qquad (2.2)$$

where G is the mass flow per unit area.

By logarithmic differentiation of Equation (2.2), we have, for quasi-steady flow,

$$dV/V = - d\rho/\rho \tag{2.3}$$

Substituting Equation (2.3) into (2.1) yields

$$\rho\, dp + G^2 4f dx/2D - G^2 d\rho/\rho = 0 \tag{2.4}$$

The equation of state for a perfect gas is,

$$\rho = p/RT \tag{2.5}$$

where R is the gas constant and T is the absolute temperature. Since the flow considered is subsonic, and thus the temperature changes are small, an isothermal process is assumed. Taking the logarithmic differential of Equation (2.5),

$$d\rho/\rho = dp/p \tag{2.6}$$

Substituting Equations (2.5) and (2.6) into Equation (2.4) yields

$$(p/RT)\frac{dp}{dx} + G^2 4f/2D - (G^2/p)\frac{dp}{dx} = 0 \tag{2.7}$$

or,

$$G^2 = \frac{(p/RT)\frac{dp}{dx}}{(1/p)\frac{dp}{dx} - 4f/2D} \tag{2.8}$$

The friction factor, 4f, is approximated empirically from Nikuradse's[8] data for smooth pipes, for turbulent flow. This was done in this study by assuming that 4f varied exponentially with Reynolds

Numbers for the range of Reynolds Numbers between 10,000 and 100,000, as shown in Fig. 1. This approximation gives,

$$4f = .271 \, Rey^{-.237} \tag{2.9}$$

But, since

$$Rey = DG/\mu \tag{2.10}$$

where μ is the coefficient of viscosity, the friction factor becomes:

$$4f = .271(DG/\mu)^{-.237} \tag{2.11}$$

It is assumed that this steady state result is valid for locally quasi-steady flow which is postulated to exist within the incremental length. Substituting Equation (2.11) into (2.8) gives the equation for mass flow in the incremental length, assuming turbulent flow:

$$G^2 = \frac{(p/RT)\ \frac{\partial p}{\partial x}}{(1/p)\ \frac{\partial p}{\partial x} - \frac{.1355}{D}\ (D/\mu)^{-.237} G^{-.237}} \tag{2.12}$$

where G and p are now functions of both x and t . Since the friction always acts to oppose the flow, the sign in the denominator of Equation (2.12) must be chosen so that G^2 is always positive. In this study, since only negative values of $\frac{\partial p}{\partial x}$ were considered, the minus sign was used exclusively.

If, on the other hand, the pressures are such that laminar flow exists, it will be assumed that the steady state result,

$$4f = 64/Rey \tag{2.13}$$

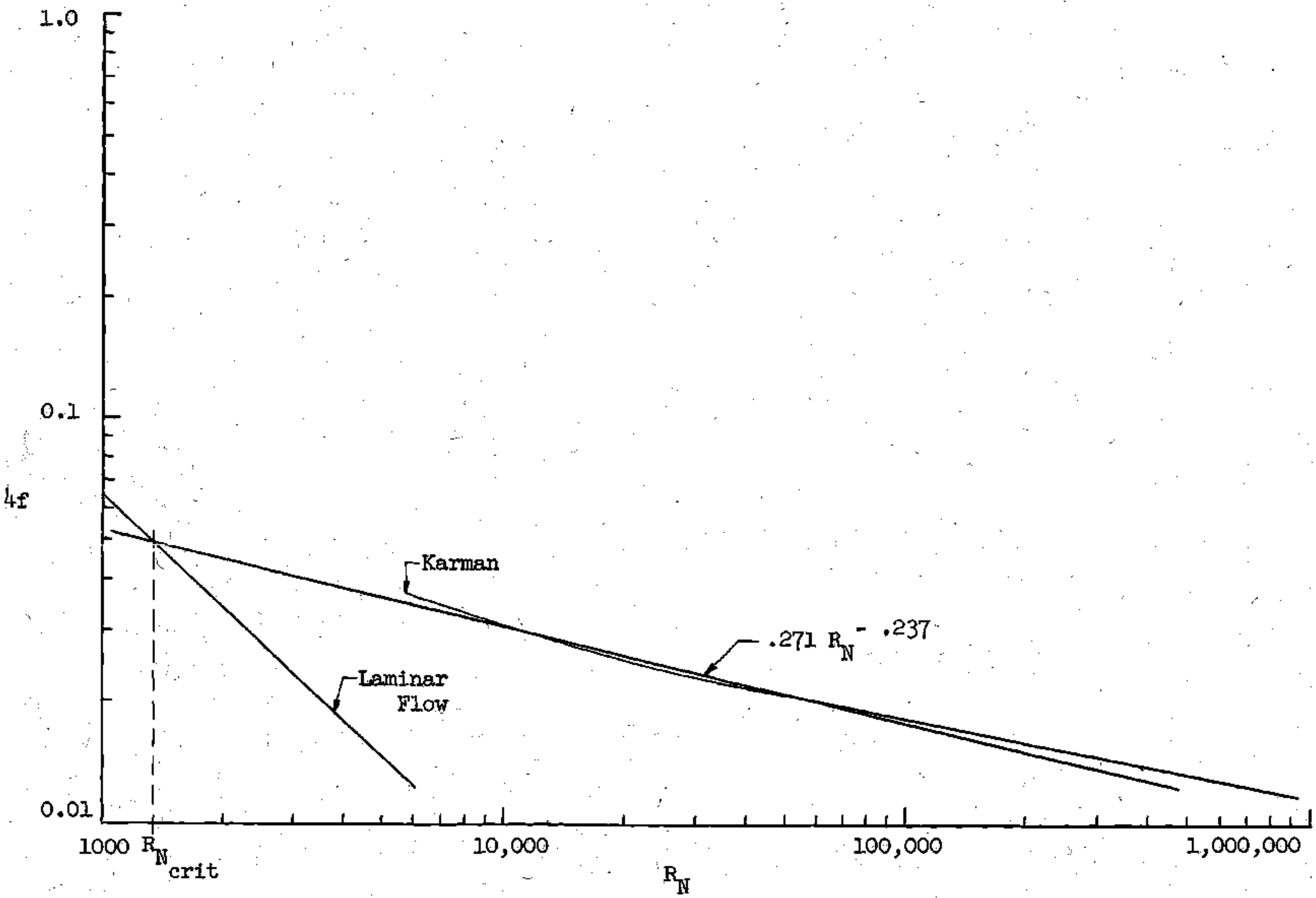

Fig. 1 Friction Factor as a Function of Reynolds Number

is applicable for this locally quasi-steady flow, and the equation for mass flow becomes

$$G = \frac{16\mu/D^2 \pm \left[(16\mu/D^2)^2 + (1/RT)(\frac{\partial p}{\partial x})^2\right]^{1/2}}{(1/p)\frac{\partial p}{\partial x}} \tag{2.14}$$

Since the denominator is always negative for a positive mass flow, G, the numerator must also be negative. Therefore, the minus sign must be chosen in Equation (2.14). The mass flow equation for laminar flow through the incremental length can now be written as

$$G = \frac{16\mu/D^2 - \left[(16\mu/D^2)^2 + (1/RT)(\frac{\partial p}{\partial x})^2\right]^{1/2}}{(1/p)\frac{\partial p}{\partial x}} \tag{2.15}$$

The equation of continuity may be written as

$$\frac{\partial G}{\partial x} = - \frac{\partial \rho}{\partial t} \tag{2.16}$$

Using the assumption of isothermal flow, and differentiating Equation (2.5) with respect to time,

$$\frac{\partial \rho}{\partial t} = (1/RT)\ \frac{\partial p}{\partial t} \tag{2.17}$$

Combining Equations (2.16) and (2.17) gives:

$$\frac{\partial G}{\partial x} = -\ (1/RT)\ \frac{\partial p}{\partial t} \tag{2.18}$$

<u>Non-Dimensionalization of Basic Equations.</u>--A set of basic parameters

governing the flow can be found by non-dimensionalizing Equations (2.12), (2.15) and (2.18). The quantities used to non-dimensionalize the variables are as follows, where the barred symbols are the non-dimensional variables:

$$\bar{p} = p/p_o \tag{2.19}$$

where p_o is the pressure at $x = 0$,

$$\bar{x} = x/L \tag{2.20}$$

where L is the length of the tube,

$$\bar{t} = tp_o/\mu \tag{2.21}$$

where is the coefficient of viscosity,

$$\bar{G} = LG/\mu \tag{2.22}$$

Using these quantities, Equation (2.12) for turbulent flow becomes:

$$\bar{G} = \left(\frac{((L^2 p_o{}^2)/(\mu^2 RT))\ (\bar{p})\ \dfrac{\partial \bar{p}}{\partial \bar{x}}}{(1/\bar{p})\ \dfrac{\partial \bar{p}}{\partial \bar{x}} - .1355\ (L/D)^{1.237}\ \bar{G}^{-.237}} \right)^{1/2} \tag{2.23}$$

Equation (2.15) for laminar flow becomes:

$$\bar{G} = 16(L/D)^2 \ \frac{1 - \left(1 + (Lp_o/\mu)^2(1/RT)(2/156)(D/L)^4(\partial \bar{p}/\partial \bar{x})^2\right)^{1/2}}{(1/\bar{p})(\partial \bar{p}/\partial \bar{x})} \tag{2.24}$$

Equation (2.18), becomes:

$$\partial \bar{G}/\partial \bar{x} = -(Lp_o/\mu)^2(1/RT)(\partial \bar{p}/\partial \bar{t}) \tag{2.25}$$

which is valid for both laminar and turbulent flow.

Equations (2.23), (2.24), and (2.25) are the basic equations used in this study. Note that two independent dimensionless parameters, L/D and $L^2p_o{}^2/\mu^2RT$, are present in the basic equations.

<u>Physical Situation</u>.--A physical situation is assumed, as shown in Fig. 2. At time $t = 0$, the pressure at the left end of the tube is p_o, and the pressure throughout the rest of the tube is the same as the pressure at the right end, p_{10}. This pressure discontinuity is sustained by a diaphragm located at the left end of the tube.

The tube is divided into 11 equally spaced stations, designated by the subscripts 0 through 10. At time $t = 0$, the diaphragm is removed, the end pressures are held fixed throughout the process, and the pressures and mass flows at the stations between are allowed to seek their equilibrium values.

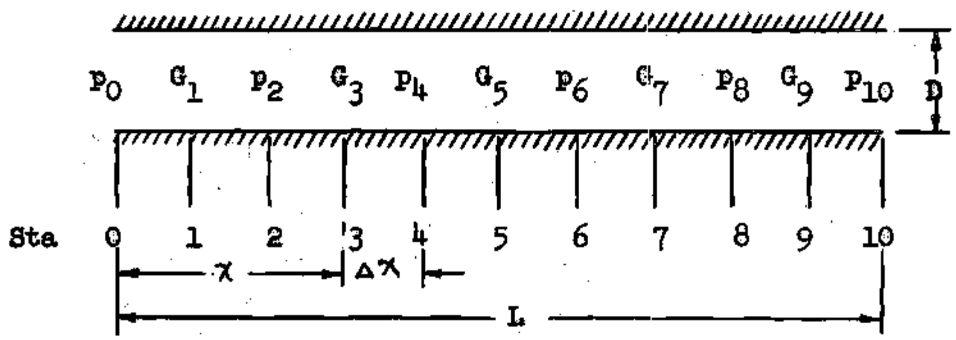

Fig. 2 Schematic of System

The assumption of quasi-steady flow within these increments is made. That is, assume G_n , the mass flow at any odd station n at any instant depends only on p_{n-1} and p_{n+1} , the end pressures for the segment of the tube enclosing station n , which are functions of time alone.

<u>Boundary Conditions for the Specific Quasi-Steady Problem</u>.--Four sets of boundary conditions were used in this study. They are as follows:

Set 1. at $t = 0$, $\bar{G}_n = 0$, $\bar{p}_o = 1$, $\bar{p}_2$ through $\bar{p}_{10} = .5$

For $t > 0$, $\bar{p}_o = 1$, $\bar{p}_{10} = .5$

Set 2. at $t = 0$, $\bar{G}_n = 0$, $\bar{p}_o = 1$, $\bar{p}_2$ through $\bar{p}_{10} = .1$

For $t > 0$, $\bar{p}_o = 1$, $\bar{p}_{10} = .1$

Set 3. at $t = 0$, $\bar{G}_n = 0$, $\bar{p}_o$ through $\bar{p}_8 = 1$, $\bar{p}_{10} = .5$

For $t > 0$, $\bar{p}_o = 1$, $\bar{p}_{10} = .5$

Set 4. at $t = 0$, $\bar{G}_n = 0$, $\bar{p}_o$ through $\bar{p}_8 = 1$, $\bar{p}_{10} = .1$

For $t > 0$, $\bar{p}_o = 1$, $\bar{p}_{10} = .1$

In conjunction with each of the above sets of boundary conditions, six cases of the governing parameters were considered, where Q is defined as:

$$Q = \frac{L^2 {p_o}^2}{\mu^2 RT} \tag{2.26}$$

	Q	L/D
Case 1	7.04×10^{13}	100
Case 2	7.04×10^{15}	100
Case 3	7.04×10^{17}	100
Case 4	7.04×10^{13}	1000
Case 5	7.04×10^{15}	1000
Case 6	7.04×10^{17}	1000

These values of Q were chosen to bracket the range of pressures from 2.0 psia to 200.0 psia for air at room temperature.

CHAPTER III

METHODS OF SOLUTION

The general solution of the system of simultaneous partial differential equations comprised of Equations (2.12), (2.15) and (2.18) was not attempted because of their non-linear character. Instead, a particular solution of the equations subject to the prescribed boundary conditions of the problem was sought. This was done by approximation of the system of non-linear partial differential equations by a system of finite difference equations. The following approximations were used for this purpose.

$$\left.\frac{\partial \bar{p}}{\partial \bar{x}}\right|_{\bar{x},\bar{t}} = \frac{\bar{p}_{\bar{x}+\Delta\bar{x},\bar{t}} - \bar{p}_{\bar{x}-\Delta\bar{x},\bar{t}}}{2\Delta\bar{x}} \qquad (3.1)$$

$$\left.\frac{\partial \bar{G}}{\partial \bar{x}}\right|_{\bar{x},\bar{t}} = \frac{\bar{G}_{\bar{x}+\Delta\bar{x},\bar{t}} - \bar{G}_{\bar{x}-\Delta\bar{x},\bar{t}}}{2\Delta\bar{x}} \qquad (3.2)$$

$$\left.\frac{\partial \bar{p}}{\partial \bar{t}}\right|_{\bar{x},\bar{t}} = \frac{\bar{p}_{\bar{x},\bar{t}+\Delta\bar{t}} - \bar{p}_{\bar{x},\bar{t}}}{\Delta\bar{t}} \qquad (3.3)$$

Dropping the bars for convenience, and substituting Equations (3.1), (3.2), and (3.3) into Equations (2.23) and (2.24), one obtains the following turbulent flow equation:

$$G_{x,t} = \left[\frac{Q(1/4\,\Delta x)(p_{x-\Delta x,t} + p_{x+\Delta x,t})(p_{x+\Delta x,t} - p_{x-\Delta x,t})}{(1/\Delta x)(p_{x+\Delta x,t} - p_{x-\Delta x,t})/(p_{x+\Delta x,t} + p_{x-\Delta x,t}) - .1355(L/D)^{1.237} G_{x,t}^{-.237}} \right]^{1/2} \tag{3.4}$$

and the laminar flow relation:

$$G_{x,t} = (16L/D)\left[\frac{1-(1+(Q/256)(D/L)^4((p_{x+\Delta x,t} - p_{x-\Delta x,t})/2\Delta x)^2)^{1/2}}{(1/\Delta x)(p_{x+\Delta x,t} - p_{x-\Delta x,t})/(p_{x+\Delta x,t} + p_{x-\Delta x,t})} \right] \tag{3.5}$$

Substitution of Equations (3.1), (3.2) and (3.3) into Equation (2.25) and re-arranging gives

$$p_{x,t+\Delta t} = p_{x,t} + (G_{x-\Delta x,t} - G_{x+\Delta x,t})\Delta N \tag{3.6}$$

for either laminar or turbulent flow, where ΔN is a non-dimensional time increment, given by

$$\Delta N = (\mu/Lp_o)^2(RT\Delta t/2\Delta x) = \Delta t/Q\Delta x \tag{3.7}$$

It should be re-emphasized at this point that all the terms appearing in Equations (3.4), (3.5) and (3.6) are non-dimensional, and it may be noted that Q, $\frac{L}{D}$, and ΔN are the non-dimensional parameters which are dictated by the prescribed boundary and initial conditions of a particular problem.

Now that the partial differential equations have been approximated by finite difference equations, and the boundary conditions and parameters established, the mechanics of solution must be considered.

The pressure distribution is known initially. From the equations for mass flow, the pressures at two adjacent stations define the mass flow between. In the laminar case, this mass flow can be found directly. However, in the turbulent case, the mass flow is only implicitly defined between the two stations at which the pressure is known. An iterative scheme was used to overcome this difficulty. It was found by trial calculations that the accuracy was increased roughly one significant digit for each iteration.

The question arises as to whether the flow is laminar or turbulent. The method used in this study to determine if a particular combination of pressures gave turbulent or laminar flow was to assume initially that the mass flow was the "critical" mass flow. This "critical" mass flow is defined in Fig. 2, as the point of intersection of the $4f = 64$ Rey curve with the $4f = .271\ \mathrm{Rey}^{-.237}$ curve for turbulent flow. This value was then used as a first approximation in the right hand side of the turbulent mass flow equation, Equation (3.4). If the mass flow resulting from calculating the right hand side was greater than that assumed, the flow was considered turbulent, and if the calculated mass flow was less than "critical", the flow was considered laminar.

The parameter ΔN , a non-dimensional time increment, must be established. The choice of the non-dimensional time increment has a very decided effect on the behavior of the step-by-step solution of Equations (3.4), (3.5) and (3.6). If the ΔN chosen is too large, the successive values of pressure or mass flow at a station found by this step-by-step method oscilliate over and under the correct solution. If, on the other hand, ΔN is too small, the calculations proceed smoothly, but very little change in values of pressure and mass flow at a station occurs until an excessive number of steps has been completed. Thus, a compromise is needed between accuracy potentially obtainable and time required for calculation. By trial computations it was found that with the initial pressure distribution a step function, the initial ΔN had to be very small, compared with the ΔN's permissible after the pressure distribution had changed from a step function to smooth function of $\bar{\bar{x}}$. On the basis of these considerations, it was decided to start with some suitably small initial time increment, and subsequently double the time increment after each ten successive steps. The choice of the size of the initial time increment was dictated by the magnitudes of the parameters Q and L/D.

CHAPTER IV

COMPUTER SOLUTION

Equations (3.4), (3.5) and (3.6) were programmed in the Runcible system for solution with the aid of the IBM 650 data processing machine, and solutions were obtained for each set of boundary and initial conditions in conjunction with each set of non-dimensional similarity parameters.

The general procedure for solution was as follows: For each case, the initial pressure distribution was fed in as data; that is, the non-dimensional pressures at stations 0, 2, 4, 6, 8, and 10. Using this data and Equations (3.7) and/or (3.8), the mass flow was determined at stations 1, 3, 5, 7, and 9, corresponding respectively to the pressures at stations 0 and 2, 2 and 4, 4 and 6, etc. These values of mass flow along with the corresponding pressures were then read out. The change in pressure at stations 2, 4, 6, and 8 was next found by considering respectively the previously obtained mass flows at stations 1 and 3, 3 and 5, etc., and using Equation (3.6). This new pressure distribution was then used to repeat the process until the mass flow at station 1 differed from the mass flow at station 9 by less than 10% of their mean value. The machine time required to find a mass flow distribution and the corresponding pressure distribution varied from 20 seconds to two minutes. This time was influenced largely by the relative magnitude of the two terms in the denominator of Equation (3.4).

To express how nearly conditions had converged to their steady state values, a "per cent converged" was used which is defined as follows:

$$\text{"\% converged"} = 100 \left[1 - \frac{2(G_1 - G_9)}{(G_1 + G_9)} \right] \qquad (4.1)$$

This "per cent converged" is merely 100 times 1 minus the difference in the mass flows at the end stations divided by their mean value. It may be noted that this quantity is not a true percentage, since it may assume any value between - 100 and 100. Also, when this quantity has a value of, say, 90, the steady state mass flow actually lies about half-way between the value of G_1 and G_9 . Thus, the deviation of G_1 and G_9 from the steady state mass flow is only about 5%, instead of the 10% indicated by the "percentage converged" figure. That is, G_1 is about 5% more than the steady state value of G , while G_9 is about 5% less than G . An interesting sidelight is that $(G_1 + G_9)$ proved to be only a weak function of time -- ie., almost constant in most cases considered.

CHAPTER V

RESULTS

The "% converged" is plotted as a function of non-dimensional time in Fig. 3 and Fig. 4, for a representative set of boundary and initial conditions. Given an initial Q, the relative effect of different boundary and initial conditions on the time required for a given degrée of convergence may be seen. The initial pressure distribution has a large effect, in that the closer this distribution is to the final distribution the less the time required for convergence to steady state conditions.

To further analyze these trends, the time required to achieve a "% converged" of 90% was used for a standard. In Fig. 5 and Fig. 6, the natural logarithm of Q is presented as a function of the natural logarithm of the non-dimensional time, using the above standard of comparison, for each initial non-dimensional pressure distribution. It may be noted at this point that the computed data were only sufficient to locate three points on each curve. The resulting curves are almost straight. The average slopes of the curves within the range of Q's considered was then obtained. From this consideration, the following approximate relationships were obtained:

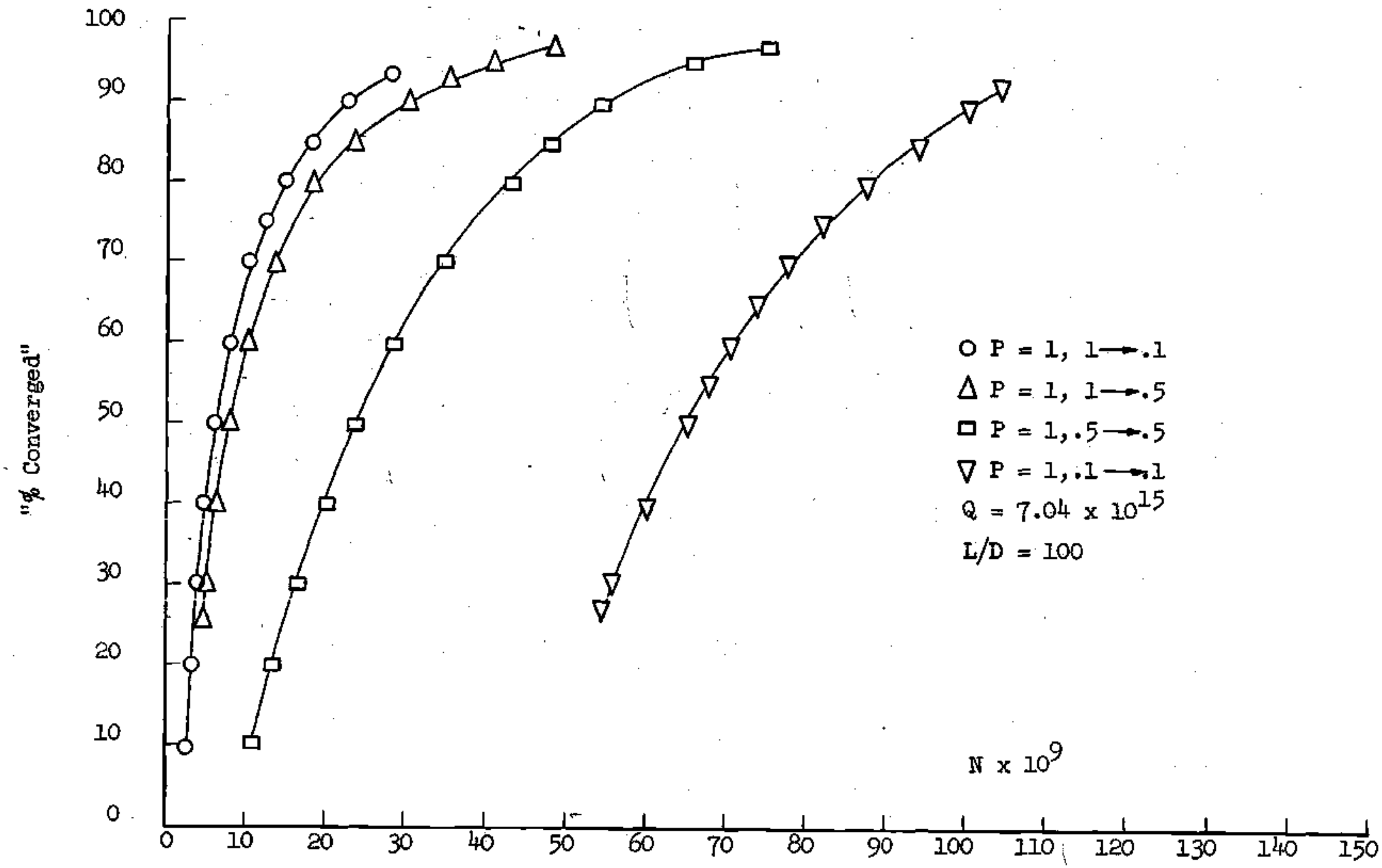

Fig. 3 Representative Plot of Non-Dimensional Time Versus "% Converged" (L/D = 100)

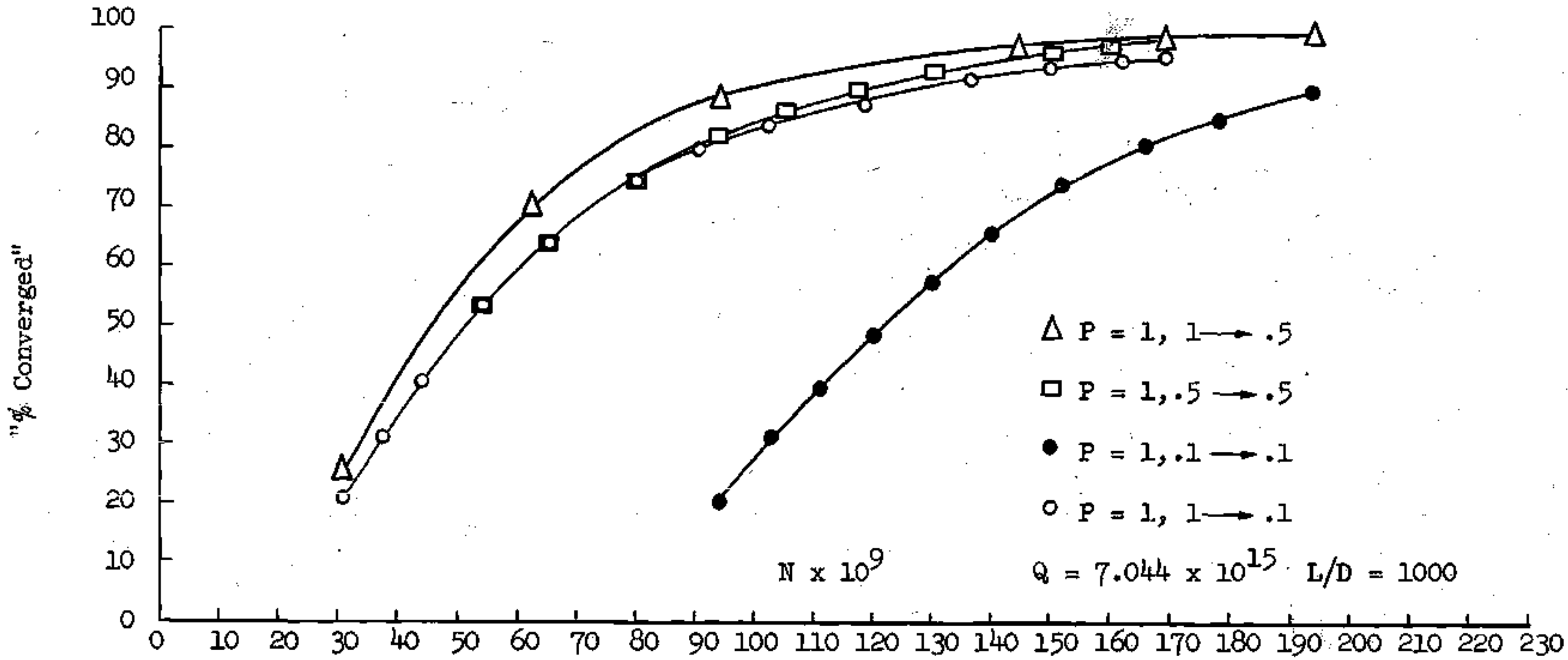

Fig. 4 Representative Plot of Non-Dimensional Time Versus "% Converged" (L/D = 1000)

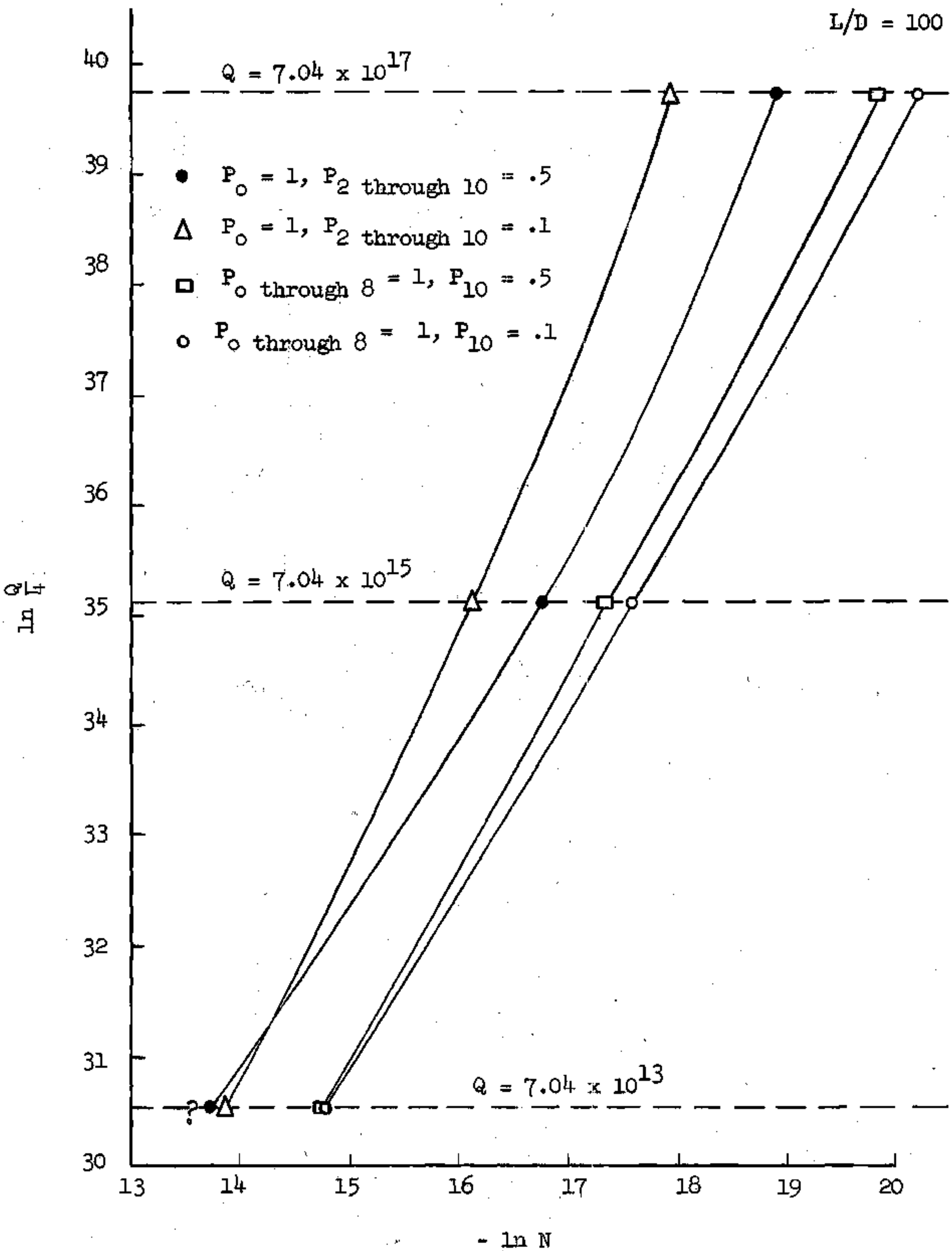

Fig. 5 Non-Dimensional Time Required for 90% Convergence

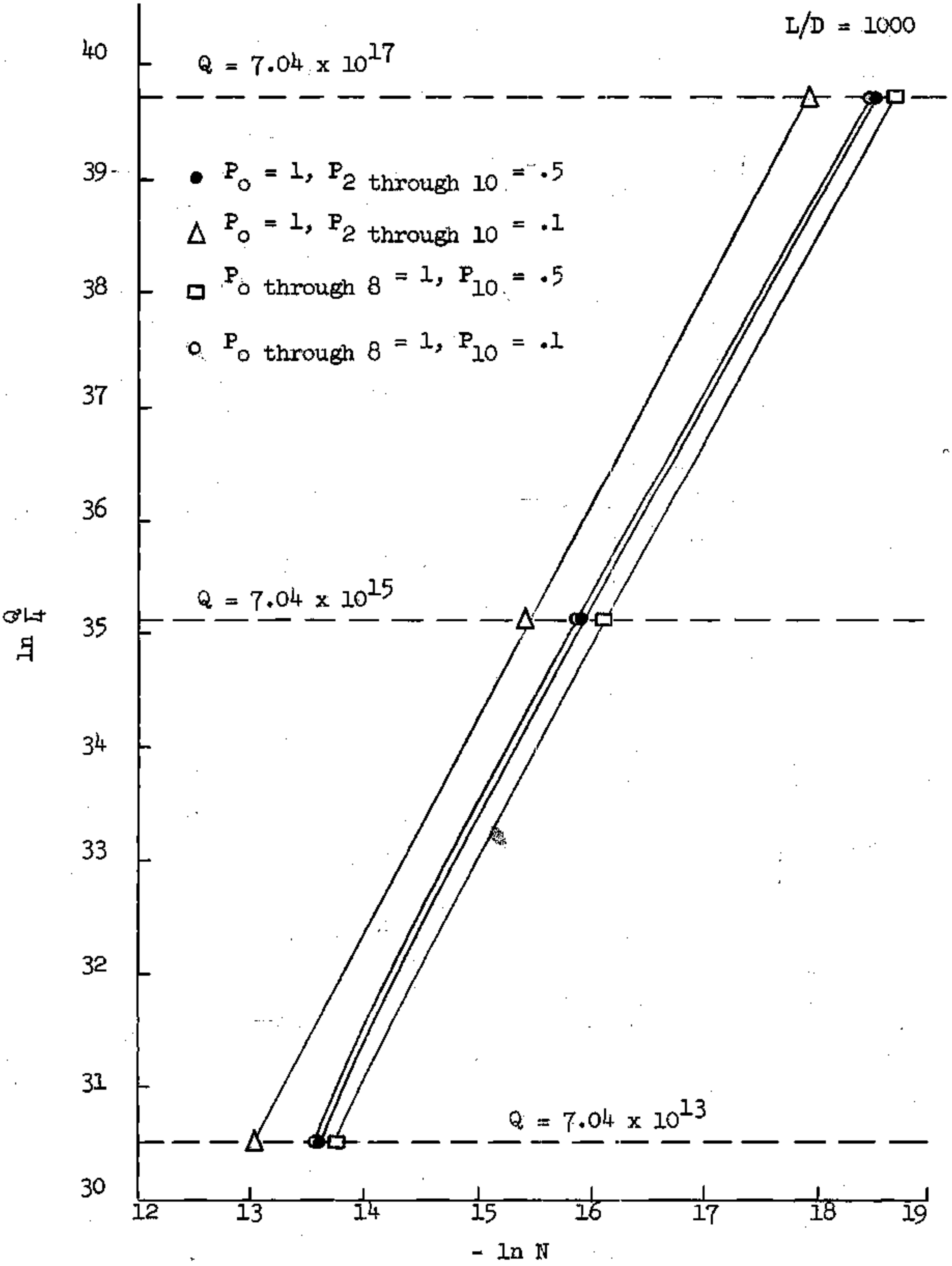

Fig. 6 Non-Dimensional Time Required for 90% Convergence

Set	$\frac{L}{D} = 100$	$\frac{L}{D} = 1000$
1	$N \propto Q^{-.550}$	$N \propto Q^{-.532}$
2	$N \propto Q^{-.443}$	$N \propto Q^{-.526}$
3	$N \propto Q^{-.553}$	$N \propto Q^{-.535}$
4	$N \propto Q^{-.595}$	$N \propto Q^{-.535}$

The slopes of the curves presented in Fig. 5 and 6 are approximately - 1/2. This indicates that N is nearly proportional to $Q^{-1/2}$.

From Equation (3.7) and Equation (2.26):

$$N = (\mu/Lp_o)^2(RT\bar{t}/2\Delta\bar{x})$$

$$Q = (L^2p_o^{\,2}/\mu^2RT)$$

Assume that the following proportionality is adequate.

$$N \propto Q^{-1/2} \qquad \text{(approximately)}$$

or

$$(\mu/Lp_o)^2(RT\bar{t}/2\Delta\bar{x}) \propto (RT)^{1/2}/Lp_o \qquad (5.1)$$

Now, $\bar{t} = tp_o/\mu$ from Equation (2.21). Combining this relation with Equation (5.1), one obtains

$$t_c \propto 2\ \bar{x}L/(RT)^{1/2} \quad \text{(approximately)} \qquad (5.2)$$

where t_c is the real time corresponding to "% converged" of 90%. Since $\bar{x}$ is the non-dimensional length of the increment considered, and thus only a number (in this case, 0.1) Equation (5.2) can be written

$$t_c \sim L/(RT)^{1/2} \quad \text{(approximately)} \qquad (5.3)$$

where t_c is the time in seconds necessary for the steady state mass flow to develop in a tube L feet in length with the fluid at $T°$ absolute. This analysis indicates that the time to converge is only a weak function of the pressures used, with the limitation, of course, that the pressures must be such that the fundamental equations which this study is based on are applicable.

To further analyze this relation, define a dimensionless parameter, η,

$$\eta = t_c(RT)^{1/2}/L \qquad (5.4)$$

Now, using Equations (2.21), (2.26) and (3.7) in conjunction with (5.4), the following relations are obtained:

$$\eta = \bar{t}_c/Q^{1/2} \equiv Q^{1/2}N/5 \qquad (5.5)$$

where $\bar{t}_c$ is dimensionless time to converge to 90%. The last relation can be used with the data obtained in this study to evaluate η. Fig. 7 and 8 show that η is a weak function of Q, although in Fig. 7 the function is rather poorly defined.

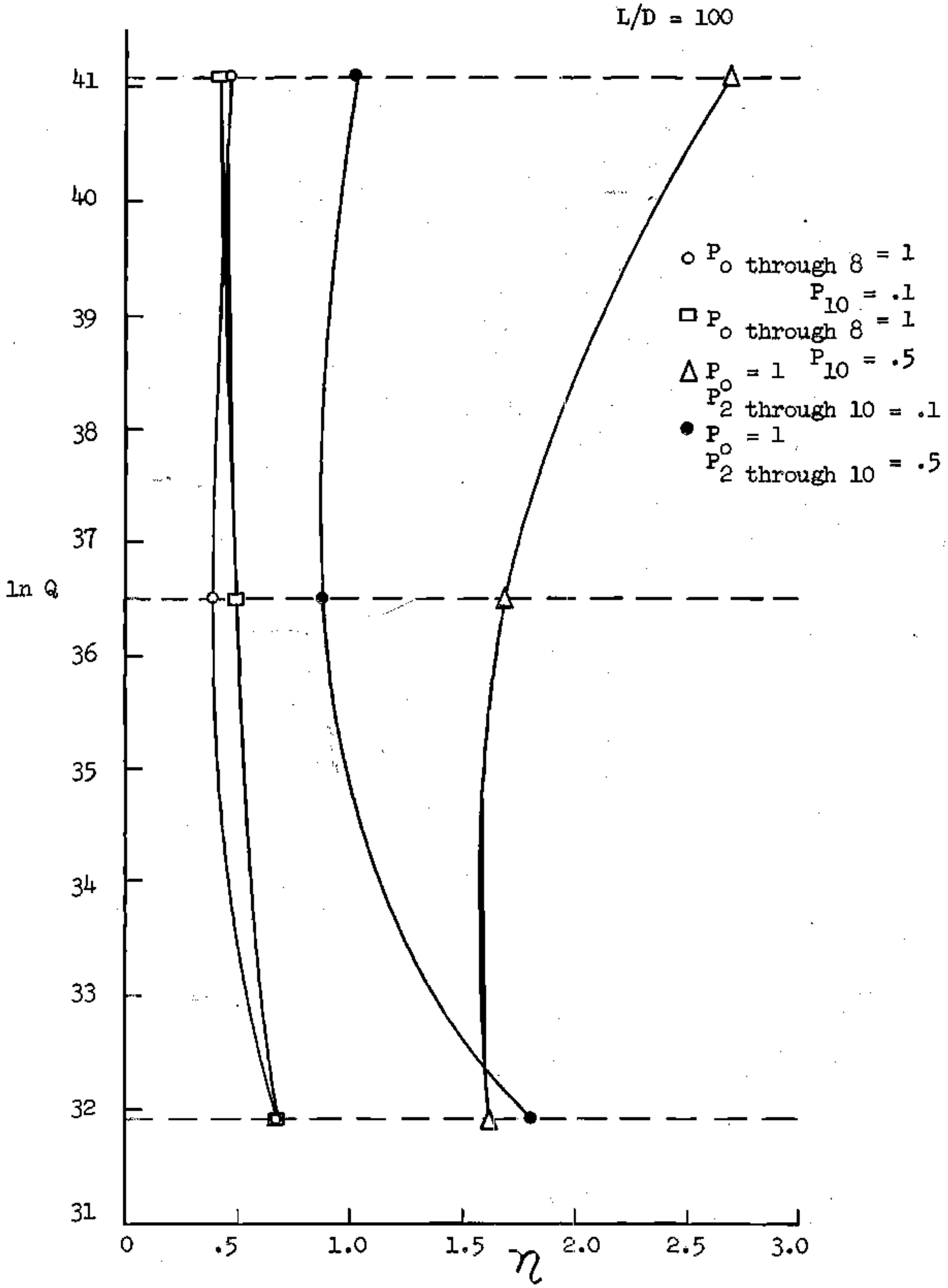

Fig. 7 Plot of lnQ as a Function of η, (L/D = 100)

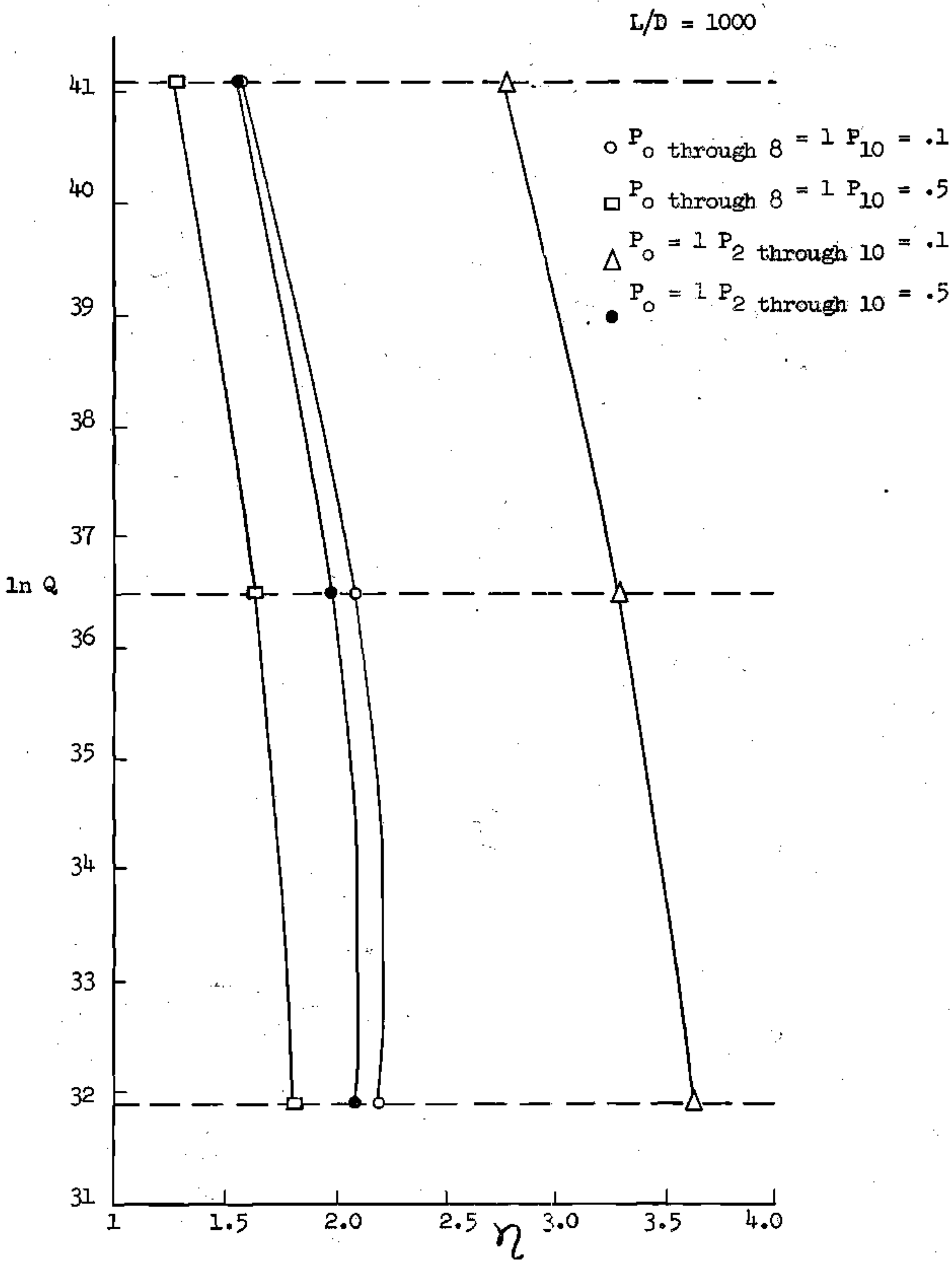

Fig. 8 Plot of lnQ as a Function of η, (L/D = 1000)

The flows produced by the conditions considered in this study are primarily turbulent, except in case IV, ($Q = 7.04 \times 10^{13}$, $L/D = 1000$). This combination of pressure level and L/D ratio gives a steady state flow which is completely laminar, with portions of the flow passing through a turbulent state as it develops. This is shown in Fig. 9.

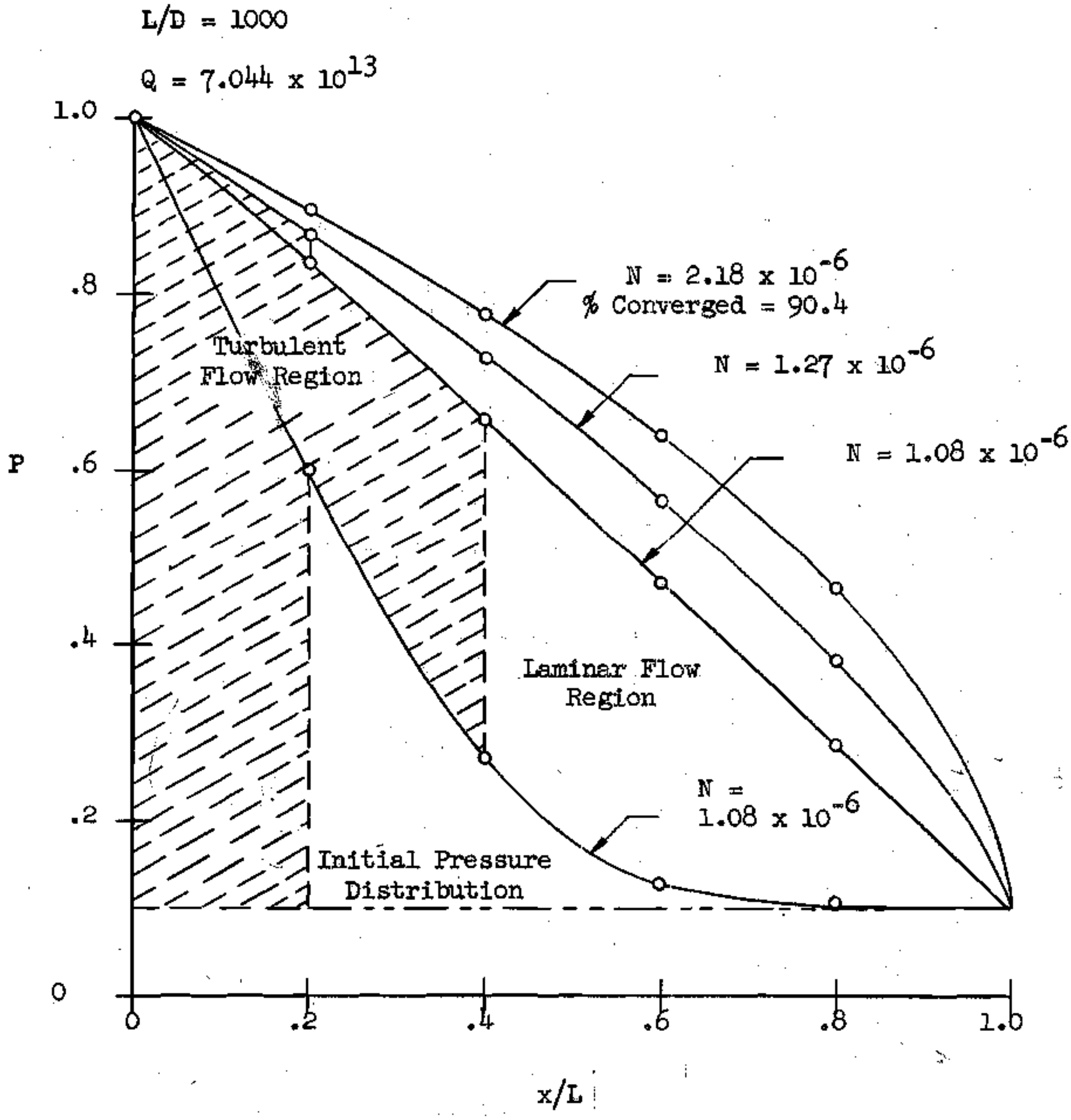

Fig. 9 Plot of Pressure Versus Station with Time as a Parameter Showing Regions of Laminar and Turbulent Flow

CHAPTER VI

CONCLUSIONS

The data presented in this study indicate that the time required for the development of steady state flow following a step input in pressure is a strong function of the initial pressure distribution and the L/D ratio of the system geometry, and only a weak function of the pressure level. This data is valid, of course, only for the range of L/D's and Q's considered in this study.

The use of this information in conjunction with the plots presented in the previous chapter facilitates the estimation of the time required for a given flow to converge to its steady state condition.

If this estimated time required for convergence is of smaller magnitude than the time element under consideration in a particular case, the assumption of quasi-steady flow should be justified for that case. Perhaps a better requirement would be that

$$\frac{dp}{dt} << \frac{p_o - p_1}{t_c}$$

CHAPTER VII

RECOMMENDATIONS

The author recommends that the values of L/D used in this study be extended so that the effect of the L/D ratio might be established on the time required for a given flow to assume its steady state condition. This proposed study might also be used to check the validity of the assumption of quasi-steady flow, which was made in this study, between the increments of the tube.

Another problem of interest would be to extend the boundary conditions considered to include end pressures that are continuously varying functions of time.

A study more pertinent to present day missile plumbing systems would be to vary the above problem by adding an initial temperature distribution imposed between the end stations of the tube.

In view of the conclusions reached in Appendix I, it is recommended that the integrated form of the differential expression for mass flow per unit area, Equation (A.4), be substituted for the finite difference approximation of this expression, Equation (3.4), in future studies of this type.

APPENDIX I

<u>Error Analysis</u>.--It is of interest to estimate the error induced by the approximation of a differential equation by a finite difference equation. For this purpose, the final steady pressure distribution is first considered. Using this condition, the properties of the flow become functions of position alone, and thus the partial derivatives can be replaced by total derivatives. Thus, Equation (2.23) becomes

$$\bar{G}^2 = \frac{Q\bar{p}\,\dfrac{d\bar{p}}{d\bar{x}}}{(1/\bar{p})\,\dfrac{d\bar{p}}{d\bar{x}} - .1355(L/D)^{1.237}\bar{G}^{-.237}} \qquad (A.1)$$

with the condition that $\bar{G} =$ constant . This Equation can be rearranged and integrated,

$$\int_{\bar{p}_{x-\Delta x}}^{\bar{p}_{x+\Delta x}} \left(\bar{G}^2 \frac{1}{\bar{p}} - Q\bar{p}\right) d\bar{p} = .1355(L/D)^{1.237}\bar{G}^{-.237}\bar{G}^2 \int_{\bar{x}-\Delta\bar{x}}^{\bar{x}+\Delta\bar{x}} d\bar{x}$$

(A.2)

It should be re-emphasized at this point that the barred quantities are non-dimensional, as is the parameter Q . Dropping the bars for convenience, and integrating Equation (A.2), one obtains the relation:

$$G^2 \ln \frac{p_{x+\Delta x}}{p_{x-\Delta x}} - \frac{Q}{2}(p^2_{x+\Delta x} - p^2_{x-\Delta x}) = .1355(\tfrac{L}{D})^{1.237} G^2 G^{-.237}(2\Delta x) \tag{A.3}$$

Solving this relation for G yields:

$$G^2 = \frac{Q(p^2_{x+\Delta x} - p^2_{x-\Delta x})}{2\left[\ln\frac{(p_{x+\Delta x})}{(p_{x-\Delta x})} - .1355\ (L/D)^{1.237}\ G^{-.237}\ (2\Delta x)\right]} \tag{A.4}$$

The finite difference approximation for this relation which was used in this study is Equation (3.4).

$$G^2 = \frac{Q(p^2_{x+\Delta x} - p^2_{x-\Delta x})}{2\left[2\frac{(p_{x+\Delta x} - p_{x-\Delta x})}{(p_{x+\Delta x} + p_{x-\Delta x})} - .1355\ (L/D)^{1.237} G^{-.237}\ (2\Delta x)\right]}$$

To facilitate notation, the G^2 obtained from Equation (A.4) is denoted by $G_E{}^2$ and the approximation of this from Equation (3.4) is denoted by G_A^2. Further, it is now assumed that the error is small, and

$$.1355(L/D)^{1.237} G_E{}^{-.237}\ (2\Delta x) \simeq .1355(L/D)^{1.237} G_A{}^{-.237}\ (2\Delta x)$$

Define for convenience,

$$B = .1355(L/D)^{1.237} G_A{}^{-.237}(2\Delta x) \tag{A.5}$$

and

$$A = Q(p^2_{x + \Delta x} - p^2_{x - \Delta x}) \tag{A.6}$$

These simplifications allow Equations (A.3) and (3.4) to be written

$$G_E^2 = \frac{A}{2\left[\ln\frac{(p_{x + \Delta x})}{(p_{x + \Delta x})} - B\right]} \tag{A.7}$$

and

$$G_A^2 = \frac{A}{2\left[2\frac{(p_{x + \Delta x} - p_{x - \Delta x})}{(p_{x + \Delta x} + p_{x - \Delta x})} - B\right]} \tag{A.8}$$

Now, the error involved may be expressed as

$$\text{Error} = \frac{G_E^2 - G_A^2}{G_E^2} = 4\ \frac{\left[\frac{p_{x + \Delta x} - p_{x - \Delta x}}{p_{x + \Delta x} + p_{x - \Delta x}}\right] - 2\ln\frac{(p_{x + \Delta x})}{(p_{x - \Delta x})}}{4\ \left[\frac{p_{x + \Delta x} - p_{x - \Delta x}}{p_{x + \Delta x} + p_{x - \Delta x}}\right] - B} \tag{A.9}$$

but,

$$\ln\left[\frac{p_{x + \Delta x}}{p_{x - \Delta x}}\right] = 2\left[\frac{p_{x + \Delta x} - p_{x - \Delta x}}{p_{x + \Delta x} + p_{x - \Delta x}} + 1/3\,\frac{(p_{x + \Delta x} - p_{x - \Delta x})^3}{(p_{x + \Delta x} + p_{x - \Delta x})} + 1/5\,\frac{(p_{x + \Delta x} - p_{x - \Delta x})^5}{(p_{x + \Delta x} + p_{x - \Delta x})} + \text{---}\right]$$

therefore, the error involved may be expressed as

$$\text{Error} = \frac{4\left[1/3\left(\frac{p_{x+\Delta x} - p_{x-\Delta x}}{p_{x+\Delta x} + p_{x-\Delta x}}\right)^3 + 1/5\left(\frac{p_{x+\Delta x} - p_{x-\Delta x}}{p_{x+\Delta x} + p_{x-\Delta x}}\right)^5\right.}{4\left[\frac{p_{x+\Delta x} - p_{x-\Delta x}}{p_{x+\Delta x} + p_{x-\Delta x}}\right] - B}$$

$$\frac{\left.+ 1/7\left(\frac{p_{x+\Delta x} - p_{x-\Delta x}}{p_{x+\Delta x} + p_{x-\Delta x}}\right)^7 + ---\right]}{4\left[\frac{p_{x+\Delta x} - p_{x-\Delta x}}{p_{x+\Delta x} + p_{x-\Delta x}}\right] - B} \tag{A.10}$$

It may be seen from Equation (A.10) that G_A always underestimates the mass flow in a segment having given end pressures. Also, the error is roughly proportional to

$$\left(\frac{p_{x+\Delta x} - p_{x-\Delta x}}{p_{x+\Delta x} + p_{x-\Delta x}}\right)^2$$

when $B = 0$ (corresponding to $G = 0$). When B assumes a value, the error further decreases, and when B becomes the dominating part of the denominator, the error becomes proportional to

$$\left(\frac{p_{x+\Delta x} - p_{x-\Delta x}}{p_{x+\Delta x} + p_{x-\Delta x}}\right)^3$$

This development emphasizes the effect of the size of the incremental lengths.

Equation (A.4) may be used to find the exact steady state pressure distribution by using the end pressures to solve for the mass flow per unit area, G , and then using this value and one end pressure to solve for an unknown pressure. This is done in Fig. 10. Also shown in Fig. 10 are the approximate steady state pressure distribution from Equation (3.4) and the pressure distribution obtained at a "% converged" of 90.

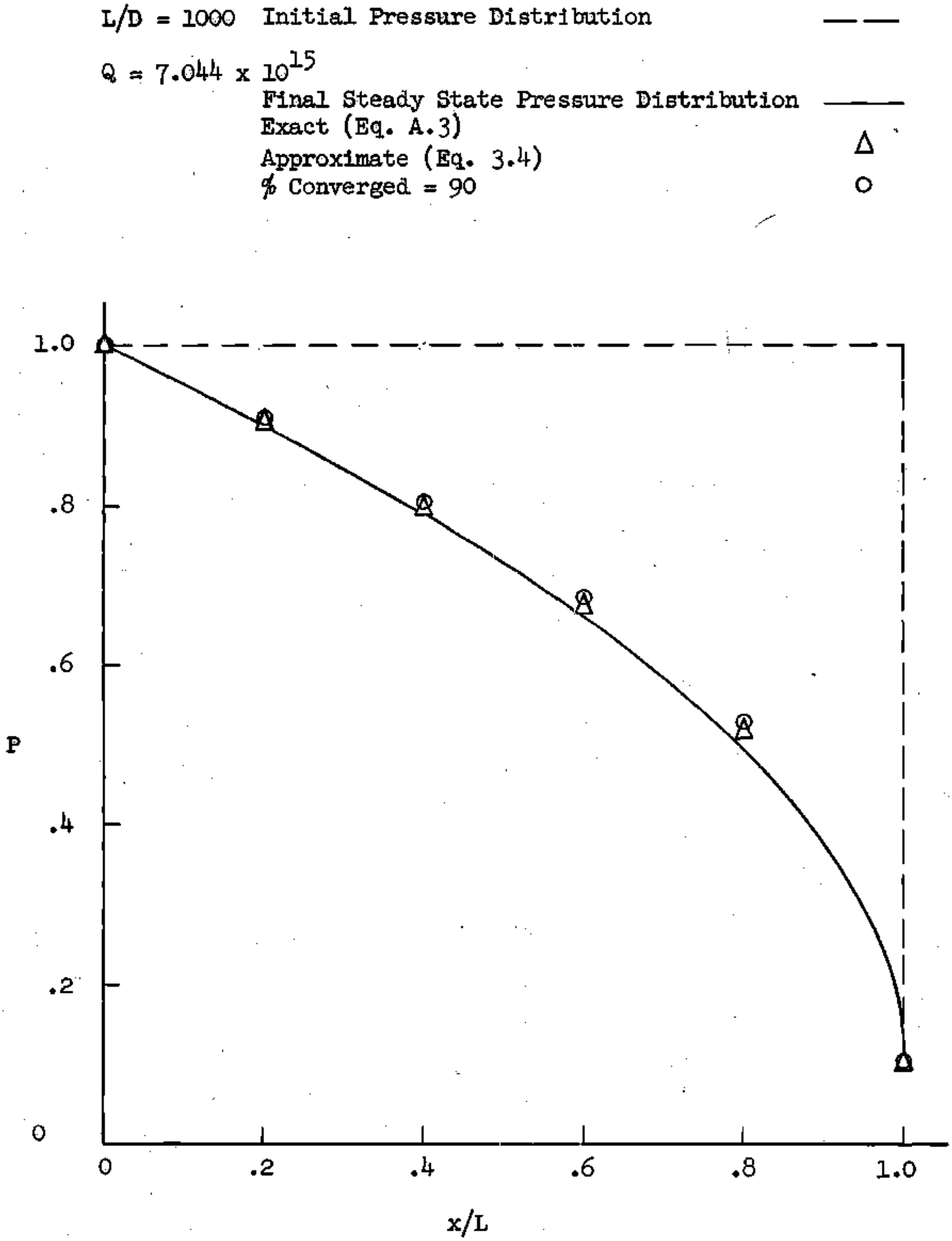

Fig. 10 Illustration of Error Due to Approximations

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